J. DIFFERENTIAL GEOMETRY
16 (1981) 293–303

GENERIC SUBMANIFOLDS OF AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

U-HANG KI & JIN SUK PAK

Dedicated to Professor Kentaro Yano on his 70th birthday

0. Introduction

Recently several authors have studied generic submanifolds (anti-holomorphic submanifolds) immersed in Kaehlerian manifolds by using the method of Riemannian fibre bundles ([3], [4] and [8] etc.).

The purpose of the present paper is to characterize generic submanifolds of an even-dimensional Euclidean space.

In §1, we recall fundamental properties and structure equations for generic submanifolds immersed in an even-dimensional Euclidean space.

In §2, we prove some lemmas under the assumption that the f-structure induced on the submanifold and the second fundamental tensors commute.

In §3, we characterize generic submanifolds of an even-dimensional Euclidean space under certain conditions.

In 1971 Yano and Ishihara [6] proved the following.

Theorem A. *Let M be a complete submanifold of dimension n immersed in a Euclidean space E^m of dimension m $(1 < n < m)$ with nonnegative sectional curvature. Suppose that the normal connection of M is flat and the mean curvature vector of M is parallel in the normal bundle. If the length of the second fundamental form of M is constant in M, then M is a sphere $S^n(r)$ of dimension n, an n-dimensional plane $E^n(\subset E^m)$, a pythagorean product of the form*

$$(1)\qquad S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N), \quad p_1 + \cdots + p_N = n, 1 < N \leqslant m - n,$$

or a pythagorean product of the form

$$(2)\qquad \begin{aligned} &S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) \times E^p, \\ &\qquad\qquad p_1 + \cdots + p_N + p = n, 1 < N \leqslant m - n, \end{aligned}$$

Communicated by K. Yano, August 11, 1980.

where $S^p(r)$ is a p-sphere with radius r, and E^p ($\subset E^m$) a p-dimensional plane. If M is a pythagorean product of the form (1) *or* (2), *then M is of essential codimension N.*

Using a method quite similar to the one used in Lemma 1.2 of Yano and Kon [8] we can prove that the sectional curvature of an n-dimensional submanifold immersed in E^m with flat normal connection is always non-negaive if the second fundamental tensor of the submanifold is parallel. By means of Theorem A, we have

Theorem B. *Let M be a complete submanifold of dimension n immersed in a Euclidean space E^m of dimension m ($1 < n < m$) with flat normal connection. If the second fundamental tensor of M is parallel, then M is of the same type as stated in Theorem A.*

To characterize the submanifolds we shall use Theorem B.

The authors would like to express here their sincere gratitude to Professor Kentaro Yano who gave them many valuable suggestions to improve the paper.

1. Structure equations of generic submanifolds

Let E^{2m} be a $2m$-dimensional Euclidean space, and 0 the origin of a Cartesian coordinate system in E^{2m}, and denote by X the position vector representing a point of E^{2m} with respect to the origin. Since E^{2m} is even-dimensional, E^{2m} can be regarded as a flat Hermitian manifold, and hence there exists a tensor field F of type (1,1) with constant components such that

$$F^2 = -I, \quad (FX)\cdot(FY) = X\cdot Y \tag{1.1}$$

for any vectors X and Y, where I denotes the identity transformation, and the dot the inner product in the Euclidean space E^{2m}.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and immersed isometrically in E^{2m} by the immersion i: $M \to E^{2m}$. Throughout this paper the indices $h, i, j, k, \cdots, t$ run over the range $\{1, 2, \cdots, n\}$, and the summation convention is used with respect to this system of indices. We identify $i(M)$ with M itself.

Put

$$X_i = \partial_i X, \quad \partial_i = \partial/\partial x^i. \tag{1.2}$$

Then X_i are n linearly independent vector fields tangent to the submanifold M. Denoting by g_{ji} the components of the induced metric tensor of M, we have

$$g_{ji} = X_j \cdot X_i, \tag{1.3}$$

since the immersion is isometric.

Denote by C_x $2m - n$ mutually orthogonal unit normals to M. Throughout this paper the indices u, v, w, x, y and z run over the range $\{n + 1, \cdots, 2m\}$, and the summation convention is used with respect to this system of indices. Therefore denoting by ∇_j the operator of the van der Waerden-Bortolotti covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^k{}_i\}$ formed with g_{ji}, we have the equations of Gauss and Weingarten for M

$$\nabla_j X_i = h_{ji}{}^x C_x, \tag{1.4}$$

$$\nabla_j C_x = -h_j{}^k{}_x X_i \tag{1.5}$$

respectively, where $h_{ji}{}^x$ are the second fundamental tensors with respect to the normals C_x and $h_j{}^i{}_x = h_{jh}{}^y g^{ih} g_{yx}$, g_{yx} being the metric tensor of the normal bundle of M given by $g_{yx} = C_y \cdot C_x$, and $(g^{ji}) = (g_{ji})^{-1}$.

Since the ambient manifold E^{2m} is Euclidean, the equations of Gauss, Codazzi and Ricci for M are respectively given by

$$K_{kji}{}^h = h_k{}^h{}_x h_{ji}{}^x - h_j{}^h{}_x h_{ki}^x, \tag{1.6}$$

$$\nabla_k h_{ji}{}^x - \nabla_j h_{ki}{}^x = 0, \tag{1.7}$$

$$K_{jiy}{}^x = h_{jt}{}^x h_i{}^t{}_y - h_{it}{}^x h_j{}^t{}_y, \tag{1.8}$$

where $K_{kji}{}^h$ and $K_{jiy}{}^x$ are the curvature tensors of M and the connection induced in the normal bundle respectively.

Now we consider the submanifold M of E^{2m} which satisfies

$$N_P(M) \perp F(N_P(M))$$

at each point $P \in M$, where $N_P(M)$ denotes the normal space at P. Such a submanifold M is called a generic submanifold (an anti-holomorphic submanifold), [4], [7]. From now on we consider generic submanifolds immersed in an even-dimensional Euclidean space E^{2m}. Then we can put in each coordinate neighborhood

$$FX_j = f_j^i X_i - f_j^x C_x, \tag{1.9}$$

$$FC_x = f_x{}^i X_i, \tag{1.10}$$

where f_j^i is a tensor field of type (1,1) defined on M, f_j^x a local 1-form for each fixed index x, and $f_x{}^i = f_j^y g^{ji} g_{yx}$.

Applying F to (1.9) and (1.10) respectively, and using (1.1) and those equations, we can easily find

$$f_j^t f_t^h = -\delta_j^h + f_j^x f_x{}^h, \tag{1.11}$$

$$f_j^t f_t^x = 0, \quad f_t^x f_j^t = 0, \tag{1.12}$$

$$(1.13)\qquad f_t^x f_y^{\ t} = \delta_y^x.$$

Moreover, (1.11) and (1.12) imply

$$f_j^h f_h^t f_t^i + f_j^i = 0,$$

and consequently M admits the so-called f-structure satisfying $f^3 + f = 0$ (see [2], [3]).

Substituting (1.9) into $(FX_j) \cdot (FX_i) = X_j \cdot X_i$ gives

$$(1.14)\qquad f_j^h f_i^k g_{hk} = g_{ji} - f_j^x f_i^y g_{xy}.$$

By putting $f_{ji} = f_j^t g_{ti}, f_{jx} = f_j^y g_{yx}$, we easily see that

$$(1.15)\qquad f_{ji} = -f_{ij}, \quad f_{jx} = f_{xj}.$$

If we apply the operator ∇_j of the covariant differentiation to (1.9) and take account of $\nabla_j F = 0$, then we obtain

$$F\nabla_j X_i = (\nabla_j f_i^h)X_h - f_i^h \nabla_j X_h - (\nabla_j f_i^x)C_x - f_i^x \nabla_j C_x.$$

Substituting (1.4) and (1.5) into the above equation yields

$$(1.16)\qquad \nabla_j f_i^h = h_{ji}{}^x f_x{}^h - h_j{}^h{}_x f_i^x,$$

$$(1.17)\qquad \nabla_j f_i^x = h_{jt}{}^x f_i^t.$$

In the same way, from (1.10) we can also obtain

$$(1.18)\qquad \nabla_j f_x{}^h = h_{jtx} f^{ht},$$

$$(1.19)\qquad f_x{}^t h_{jt}{}^y = h_j{}^t{}_x f_t^y,$$

where $h_{jtx} = h_j{}^i{}_x g_{it}$ and $f^{ht} = f_j^t g^{jh}$ because of (1.4) and (1.5).

We now consider a tensor field S of type (1,2) whose local components are given by

$$S_{ji}{}^h = [f, f]_{ji}{}^h + (\nabla_j f_i^x - \nabla_i f_j^x) f_x{}^h,$$

where

$$[f, f]_{ji}{}^h = f_j^t \nabla_t f_i^h - f_i^t \nabla_t f_j^h - (\nabla_j f_i^t - \nabla_i f_j^t) f_t^h$$

is the Nijenhuis tensor formed with f_i^h. When the tensor field S vanishes identically, the f-structure induced on M is said to be *normal* (see Nakagawa [2]). But, for generic submanifolds of a Euclidean space, substituting (1.16) and (1.17) into the above equation, we find

$$S_{ji}{}^h = (h_j{}^t{}_x f_t^h - f_j^t h_t{}^h{}_x) f_i^x - (h_i{}^t{}_x f_t^h - f_i^t h_t{}^h{}_x) f_j^x.$$

Hence if $S_{ji}{}^h$ vanishes identically, we have

$$(1.20)\qquad (h_{itx} f_h{}^t + h_{htx} f_i^t) f_j^x - (h_{jtx} f_h{}^t + h_{htx} f_j^t) f_i^x = 0,$$

because f_{ji} is skew-symmetric.

Transvecting (1.20) with $f_y{}^j$ and taking account of (1.12) and (1.13), we find

$$h_{ity}f_h{}^t + h_{hty}f_i^t - (h_{jtx}f_h{}^tf_y{}^j)f_i^x = 0. \tag{1.21}$$

Taking the skew-symmetric part with respect to the indices i and h in (1.21) yields

$$(h_{jtx}f_h{}^tf_y{}^j)f_i^x - (h_{jtx}f_i{}^tf_y{}^j)f_h{}^x = 0,$$

which, transvected with $f_z{}^i$, gives $h_{jtz}f_h{}^tf_y{}^j = 0$ because of (1.12) and (1.13). Consequently (1.21) becomes $h_{ity}f_h{}^t + h_{hty}f_i^t = 0$. Thus we have

Lemma 1.1. *Let M be an n-dimensional generic submanifold of an even-dimensional Euclidean space E^{2m}. Then the f-structure induced on M is normal if and only if*

$$h_j{}^t{}_xf_t{}^i = f_j{}^th_t{}^i{}_x. \tag{1.22}$$

Here we first notice that the condition (1.22) does not depend on the choice of mutually orthogonal unit normal vectors C_x. In fact, if we take another set of mutually orthogonal unit normals $'C_x$, then we have

$$'C_x = \sigma_x{}^yC_y, \tag{1.23}$$

where $(\sigma_x{}^y)$ is a special orthogonal matrix of degree $2m - n$. Defining the second fundamental tensor $'h_{ji}{}^x$ with respect to $'C_x$ by $\nabla_j X_i = 'h_{ji}{}^x{}'C_x$, we have,

$$'h_{ji}{}^x = \sigma_y{}^xh_{ji}{}^y,$$

which implies our assertion.

In this point of view we shall investigate some properties concerning the f-structure induced on M satisfying (1.22) for later uses.

2. Lemmas concerning $h_j{}^t{}_xf_t{}^i = f_j{}^th_t{}^i{}_x$.

In this section, we assume throughout that the f-structure induced on M satisfies (1.22), and the normal connection of M is flat. Then from (1.22) we have

$$h_{jt}{}^xf_i^t + h_{it}{}^xf_j^t = 0, \tag{2.1}$$

$$h_{jt}{}^xh_i{}^t{}_y - h_{it}{}^xh_j{}^t{}_y = 0, \tag{2.2}$$

which is a direct consequence of the equation (1.8) of Ricci.

Transvecting (2.1) with $f_k{}^i$ and taking account of (1.11), we obtain

$$h_{jk}{}^x - (h_{jt}{}^xf_y{}^t)f_k{}^y + h_{st}{}^xf_j^tf_k{}^s = 0.$$

Taking the skew-symmetric part with respect to j and k in the above equation gives

$$(h_{jt}{}^xf_y{}^t)f_k{}^y - (h_{kt}{}^xf_y{}^t)f_j^y = 0.$$

Transvecting this equation with $f_z{}^h$ we find

(2.3) $$h_{jt}{}^x f_y{}^t = P_{yz}{}^x f_j{}^z,$$

where we have put

(2.4) $$p_{yz}{}^x = h_{ji}{}^x f_y{}^j f_z{}^i.$$

Let $P_{yzx} = g_{wx} P_{yz}{}^w$. Then P_{yzx} is symmetric for all indices because of (1.19) and (2.3).

Next, transvecting (2.2) with $f_z{}^j$ and using (2.3), we can get

$$P_{zu}{}^x P_{yw}{}^u f_i^w = P_{zy}{}^u P_{uw}{}^x f_i{}^w,$$

which together with (1.13) gives

(2.5) $$P_{uz}{}^x P_{yw}{}^u = P_{uw}{}^x P_{yz}{}^u,$$

because P_{yzx} is symmetric for all indices. From (2.5) it follows that

(2.6) $$P_{uz}{}^x P_{yx}{}^u = P_x P_{yz}{}^x,$$

where we have put

(2.7) $$P^x = g^{yz} P_{yz}{}^x.$$

Lemma 2.1. *Let M be a generic submanifold of an even-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M satisfies* (1.22), *then we have*

(2.8) $$h_{jt}{}^x h_i{}^t{}_y = P_{yz}{}^x h_{ji}{}^z.$$

Proof. Differentiating (2.3) covariantly along M and using (1.17), we find

$$(\nabla_k h_{jt}{}^x) f_y{}^t + h_j{}^{tx} h_{ksy} f_t{}^s = (\nabla_k P_{yz}{}^x) f_j^z + P_{yz}{}^x h_{kt}{}^z f_j{}^t.$$

Taking the skew-symmetric part in the above equation and using (1.7) and (2.1), we obtain

(2.9) $$2h^{stx} h_{ksy} f_{jt} = (\nabla_k P_{yz}{}^x) f_j^z - (\nabla_j P_{yz}{}^x) f_k{}^z + 2P_{yz}{}^x h_{kt}{}^z f_j{}^t.$$

Transvecting (2.9) with $f_w{}^j$ gives

(2.10) $$\nabla_k P_{yw}{}^x = (\nabla_t P_{yz}{}^x) f_w{}^t f_k{}^z,$$

which implies

$$(\nabla_k P_{yz}{}^x) f_j^z = f_y{}^t (\nabla_t P_{zw}{}^x) f_k{}^w f_j{}^z,$$

since $P_{yz}{}^x = P_{zy}{}^x$. Therefore (2.9) reduces to

$$h_t{}^{sx} h_{ksy} f_j^t = P_{yz}{}^x h_{kt}{}^z f_j{}^t,$$

Transvecting the above equation with f_i^j and taking account of (1.11), we obtain

$$h_i{}^{sx} h_{ksy} + h_t{}^{sx} h_{ksy} f_i{}^w f_w{}^t = P_{yz}{}^x h_{ki}{}^z + P_{yz}{}^x h_{kt}{}^z f_i{}^w f_w{}^t,$$

which together with (2.3) implies

$$h_i^{sx}h_{ksy} + P_{wz}{}^xP_{uy}{}^zf_k{}^uf_i{}^w = P_{yz}{}^xh_{ki}{}^z + P_{yz}{}^xP_{wu}{}^zf_k{}^uf_i{}^w.$$

Thus (2.8) is verified with the help of (2.5), and consequently the proof of the lemma is completed.

Lemma 2.2. *Under the same assumptions as those stated in Lemma* 2.1, *we have*

$$\nabla_j h^x = \nabla_j P^x, \tag{2.11}$$

where $h^x = g^{ji}h_{ji}{}^x$.

Proof. Differentiating (2.1) covariantly and using (1.16), we find

$$(\nabla_k h_{jt}{}^x)f_i{}^t + h_{jt}{}^x\big(h_{ki}{}^yf_y{}^t - h_k{}^t{}_yf_i{}^y\big) + (\nabla_k h_{it}{}^x)f_j{}^t + h_{it}{}^x\big(h_{kj}{}^yf_y{}^t - h_k{}^t{}_yf_j{}^y\big) = 0,$$

which together with (2.3) and (2.8) implies

$$(\nabla_k h_{jt}{}^x)f_i{}^t + (\nabla_k h_{it}{}^x)f_j{}^t = 0.$$

By taking the skew-symmetric part of the above equation with respect to the indices k and j, we see that

$$(\nabla_k h_{it}{}^x)f_j{}^t - (\nabla_j h_{it}{}^x)f_k{}^t = 0.$$

The last two equations together with (1.7) give $(\nabla_k h_{it}{}^x)f_j{}^t = 0$. Transvecting this equation with $f_l{}^j$ and using (1.11) we obtain

$$\nabla_k h_{il}{}^x = (\nabla_k h_{it}{}^x)f_l{}^yf_y{}^t,$$

which transvected with g^{il} thus yields

$$\nabla_k h^x = (\nabla_k h_{it}{}^x)f^{iy}f_y{}^t. \tag{2.12}$$

On the other hand, from (2.4) and (2.7) we have

$$P^x = h_{st}{}^xf^{sy}f_y{}^t.$$

If we differentiate the above equation covariantly and take account of (2.12), then we have

$$\nabla_j P^x = \nabla_j h^x + h_{st}{}^x(\nabla_j f^{sy})f_y{}^t + h_{st}{}^xf^{sy}\big(\nabla_j f_y{}^t\big).$$

Substituting (1.18) into the above equation and using (1.12), we arrive at (2.11). Hence Lemma 2.2 is proved.

3. Some characterizations of generic submanifolds

We first prove

Lemma 3.1. *Let M be a generic submanifold of an even-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on*

M satisfies (1.22), *then we have*

$$(3.1)\qquad \frac{1}{2}\Delta\left(h_{ji}{}^{x}h^{ji}{}_{x}\right) = (\nabla_j\nabla_i h^x)h^{ji}{}_x + \|\nabla_k h_{ji}{}^x\|^2,$$

where $\Delta = g^{ji}\nabla_j\nabla_i$.

Proof. From the Ricci identity and (1.8) and $K_{jiy}{}^{x} = 0$:

$$\nabla_k\nabla_j h_{ih}{}^x - \nabla_j\nabla_k h_{ih}{}^x = -K_{kji}{}^t h_{th}{}^x - K_{kjh}{}^t h_{it}{}^x,$$

we obtain, in consequence of (1.7),

$$(3.2)\qquad \nabla^k\nabla_k h_{ji}{}^x - \nabla_j\nabla_h h^x = K_{jt}h_i{}^{tx} - K_{kjih}h^{khx}$$

where K_{ji} is the Ricci tensor of M given by

$$(3.3)\qquad K_{ji} = h^x h_{jix} - h_{jt}{}^x h_i{}^t{}_x.$$

Transvecting (3.2) with $h^{ji}{}_x$ and making use of (1.6), (2.8), (3.3), (2.2) and (2.7), we get

$$(3.4)\quad \left(\nabla^k\nabla_k h_{ji}{}^x\right)h^{ji}{}_x - (\nabla_j\nabla_h h^x)h^{ji}{}_x = \left(P_{yxz}P_w{}^{yz}P_u{}^{xw} - P^y P_{yxw}P_u{}^{xw}\right)h^u.$$

Consequently (3.4) reduces to

$$\left(\nabla^k\nabla_k h_{ji}{}^x\right)h^{ji}{}_x = (\nabla_j\nabla_i h^x)h^{ji}{}_x$$

because of (2.6).

On the other hand, we have by definition

$$\frac{1}{2}\Delta\left(h_{ji}{}^x h^{ji}{}_x\right) = \left(\nabla^k\nabla_k h_{ji}{}^x\right)h^{ji}{}_x + \|\nabla_k h_{ji}{}^x\|^2.$$

Thus the last two equations give (3.1). This completes the proof of the lemma.

The mean curvature vector

$$H = \frac{1}{n}h^x C_x,$$

which is globally defined on M, is said to be parallel in the normal bundle if $\nabla_j h^x = 0$. In this case we have $\nabla_j P^x = 0$ by means of (2.11). Since $h_{ji}{}^x h^{ji}{}_x = P_x h^x$, the function $h_{ji}{}^x h^{ji}{}_x$ is constant on M. Hence (3.1) implies $\nabla_k h_{ji}{}^x = 0$, and consequently by means of Theorem B in §0 we have

Theorem 3.2. *Let M be an n-dimensional complete generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M satisfies* (1.22), *and the mean curvature vector is parallel in the normal bundle, then M is an n-sphere $S^n(r)$, an n-dimensional plane E^n ($\subset E^{2m}$), a pythagorean product of the form*

(1) $S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N)$,

$$p_1, \cdots, p_N \geqslant 1, p_1 + \cdots + p_N = n,\ 1 < N \leqslant 2m - n,$$

or a pythagorean product of the form

(2) $S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) \times E^p$,

$$p_1, \cdots, p_N, p \geqslant 1, p_1 + \cdots + p_N + p = n, 1 < N \leqslant 2m - n,$$

where $S^p(r)$ is a p-sphere with radius $r > 0$ and E^p a p-dimensional plane. If M is a pythagorean product of the form (1) *or* (2), *then M is of essential codimension N.*

Combining Lemma 1.1 and Theorem 3.2 we conclude

Theorem 3.3. *Let M be an n-dimensional complete generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M is normal, and the mean curvature vector is parallel in the normal bunde, then M is of the same type as stated in Theorem* 3.2.

We next prove

Lemma 3.4. *Under the same assumptions as those stated in Lemma* 3.1, *the scalar curvature of M is constant.*

Proof. From (2.10) we have, in consequence of (2.7),

$$\nabla_i P_x = (f_x{}^t \nabla_t P_z) f_i^z \tag{3.5}$$

which implies

$$f_j^t \nabla_t P_x = 0. \tag{3.6}$$

Differentiating (3.5) covariantly and using (1.17) we find

$$\nabla_j \nabla_i P_x = \nabla_j (f_x{}^t \nabla_t P_z) f_i^z + (f_x{}^t \nabla_t P_z) h_{js}{}^z f_i^s.$$

Taking the skew-symmetric part with respect to j and i in the above equation and using (2.1) and (2.2), we obtain

$$\nabla_j (f_x{}^t \nabla_t P_z) f_i^z - \nabla_i (f_x{}^t \nabla_t P_z) f_j^z + 2(f_x{}^t \nabla_t P_z) h_{js}{}^z f_i^s = 0.$$

Transvecting the above equation with f^{ji} and using (1.11) and (1.12) give

$$(f_x{}^t \nabla_t P_z) h_{js}{}^z (-g^{sj} + f^{sy} f_y{}^j) = 0,$$

which together with (2.4) and (2.7) implies

$$(f_x{}^t \nabla_t P_z)(h^z - P^z) = 0.$$

Transvecting the above equation with f_j^x and using (1.11) and (3.6), we have $(\nabla_j P_x)(h^x - P^x) = 0$. Thus from (2.11) it follows that

$$(\nabla_j h_x)(h^x - P^x) = 0. \tag{3.7}$$

On the other hand, the scalar curvature K of M is given by

$$K = (h^x - P^x) h_x \tag{3.8}$$

because of (2.8) and (3.3). Differentiating (3.8) covariantly and taking account of (2.11) and (3.7), we can see that K is constant on M. Thus Lemma 3.4 is proved.

Finally we prove

Theorem 3.5. *Let M be an n-dimensional compact generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M satisfies* (1.22), *then M is locally symmetric.*

Proof. From (2.8) and (3.3), we have

$$K_{ji} = (h^x - P^x)h_{jix}. \tag{3.9}$$

Differentiating (3.9) covariantly and taking account of (2.11), we find

$$\nabla^k \nabla_k K_{ji} = (h^x - P^x)\nabla^k h_{jix}. \tag{3.10}$$

Substituting (1.6) and (3.9) into (3.2) and using (2.8), we obtain

$$\nabla^k \nabla_k h_{ji}{}^x - \nabla_j \nabla_i h^x = 0.$$

Thus (3.10) becomes

$$\nabla^k \nabla_k K_{ji} = (h^x - P^x)\nabla_j \nabla_i h_x = \nabla_j \nabla_i K$$

because of (2.11) and (3.8). From Lemma 3.4 it follows that $\nabla^k \nabla_k K_{ji} = 0$. Since M is compact, the identity

$$\frac{1}{2}\Delta\left(K_{ji}K^{ji}\right) = \left(\nabla^k \nabla_k K_{ji}\right)K^{ji} + \|\nabla_k K_{ji}\|^2$$

gives

$$\nabla_k K_{ji} = 0. \tag{3.11}$$

On the other hand, if we substitute (1.6) into the right-hand side of the Ricci identity:

$$\nabla_l \nabla_m K_{kjih} - \nabla_m \nabla_l K_{kjih} = K_{mlk}{}^t K_{tjih} + K_{mlj}{}^t K_{ktih} + K_{mli}{}^t K_{kjth} + K_{mlh}{}^t K_{kjit}$$

and use (2.8), then we get

$$\nabla_l \nabla_m K_{kjih} = \nabla_m \nabla_l K_{kjih},$$

which implies that

$$\nabla^l \nabla_m K_{ljih} = \nabla_m \nabla^l K_{ljih}, \tag{3.12}$$

$$\nabla^l \nabla_m K_{klih} = -\nabla_m \nabla^l K_{lkih}. \tag{3.13}$$

By means of (3.11) and the second Bianchi identity:

$$\nabla_l K_{kjih} + \nabla_k K_{jlih} + \nabla_j K_{lkih} = 0, \tag{3.14}$$

we have $\nabla^l K_{ljih} = 0$. Thus (3.12) and (3.13) reduce respectively to

$$\nabla^l \nabla_m K_{ljih} = 0, \quad \nabla^l \nabla_m K_{klih} = 0,$$

which together with (3.14) imply that

$$\nabla^l \nabla_l K_{kjih} = 0. \tag{3.15}$$

Since M is compact, from the identity:

$$\frac{1}{2}\Delta\left(K_{kjih}K^{kjih}\right) = \left(\nabla^l\nabla_l K_{kjih}\right)K^{kjih} + \|\nabla_l K_{kjih}\|^2,$$

it follows that $\nabla_k K_{ljih} = 0$ because of (3.15). This gives the proof of the theorem.

Combining Lemma 1.1 and Theorem 3.5 we have

Theorem 3.6. *Let M be an n-dimensional compact generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M is normal, then M is locally symmetric.*

Bibliography

[1] B. Y. Chen, *Geometry of submanifolds*, Marcel Dekker, New York, 1973.

[2] H. Nakagawa, *On framed f-manifolds*, Kōdai Math. Sem. Rep. **18** (1966), 293–306.

[3] J. S. Pak, *Note on anti-holomorphic submanifolds of real codimension of a complex projective space*, to appear in Kyungpook Math. J.

[4] M. Okumura, *Submanifolds of real codimension of a complex projective space*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **4** (1975) 544–555.

[5] K. Yano, *On a structure defined by a tensor field f of type* (1,1) *satisfying* $f^3 + f = 0$, Tensor **14** (1963) 99–109.

[6] K. Yano & S. Ishihara, *Submanifolds with parallel mean curvature vector*, J. Differential Geometry **6** (1971) 95–118.

[7] ——, *The f-structure induced on submanifolds of complex and almost complex space*, Kōdai Math. Sem. Rep. **18** (1966) 120–160.

[8] K. Yano & M. Kon. *Generic submanifolds*, to appear in Ann. Mat. Pura Appl.

KYUNGPOOK UNIVERSITY, TAEGU, KOREA